2 Metric geometry

At this level there are two fundamental approaches to the type of geometry we are studying. The first, called the synthetic approach, involves deciding what are the important properties of the concepts you wish to study and then defining these concepts axiomatically by their properties. This approach was used by Euclid in his Elements (around 300 B.C.E.) and was made complete and precise by the German mathematician David Hilbert (1862-1943) in his book Grundlagen der Geometrie [1899; 8th Edition 1956; Second English Edition 1921].

The second approach, called the metric approach, is due to the American mathematician, George David Birkhoff (1884-1944) in his paper "A Set of Postulates for Plane Geometry Based on Scale and Protractor" [1932]. In this approach, the concept of distance (or a metric) and angle measurement is added to that of an incidence geometry to obtain basic ideas of betweenness, line segments, congruence, etc. Such an approach brings some analytic tools (for example, continuity) into the subject and allows us to use fewer axioms.

A third approach, championed by Felix Klein (1849-1925), has a very different flavour that of abstract algebra–and is more advanced because it uses group theory. Klein felt that geometry should be studied from the viewpoint of a group acting on a set. Concepts that are invariant under this action are the interesting geometric ideas. See Millman [1977] and Martin [1982].

Definition (distance function)
A distance function on a set $\mathcal{S}$ is a function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that for all $P, Q \in \mathcal{S}$
(i) $d(P,Q) \geq 0$; (ii) $d(P,Q) = 0$ if and only if $P = Q$; and (iii) $d(P,Q) = d(Q,P)$.

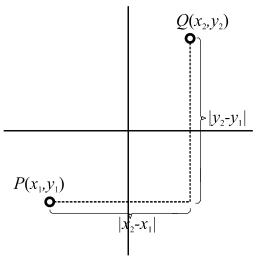
1. Let $\mathcal{M}$ denote non-empty set and let $d_M : \mathcal{M} \times \mathcal{M} \to \{0,1\}$ denote function on $\mathcal{M} \times \mathcal{M}$ defined on the following way: $\forall P, Q \in \mathcal{M}$

$$d_M(P,Q) = \begin{cases} 1, & \text{if } P \neq Q \\ 0, & \text{if } P = Q \end{cases}$$

Check is it d_M a distance function.[d_M is dist fun]

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$, and let $d_{max} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote a function which is defined on the following way $d_{max}(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Check is it d_{max} a distance function. [d_{max} is dist fun]

3. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$, and let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote a function which is defined on the following way $d(P,Q) = \sqrt{(x_1 - x_2)^2 + 4(y_1 - y_2)^2}$. Check is it d a distance function. [d is dist. fun.]



Definition (taxicab distance)
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$. The taxicab distance between P and Q is given by

$$d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|.$$

4. Show that the taxicab distance is a distance function on $\mathbb{R}^2$.

5. If d_0 and d_1 are distance functions on $\mathcal{S}$, prove that if $s \geq 0$ and $t > 0$, then $sd_0 + td_1$ is also a distance function on $\mathcal{S}$.

6. Let d denote distance function on $\mathcal{S}$, and define function $d' : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ on the following way: $\forall P, Q \in \mathcal{S}$

$$d'(P,Q) = \frac{d(P,Q)}{1 + d(P,Q)}.$$

Show that d' is also distance function on $\mathcal{S}$. Notice that $0 \leq d'(P,Q) < 1$ $\forall P, Q \in \mathcal{S}$.

Definition (surjective function)
A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a surjection if it is onto.

Remark: A function f is onto if $\forall y\ \exists x\ (f(x) = y)$, where the universe of discourse for x is the domain of the function and the universe of discourse for y is the codomain of the function.

7. Let $\mathbb{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote Cartesian plane, and let $L_{m,n}$ denote non-vertical line. Lets define function $f : L_{m,b} \to \mathbb{R}$ on the following way; $f(P) = f((x,y)) = x\sqrt{1+m^2}$; where $P \in L_{m,b}$, $P(x,y)$. Show that f is surjection. $[P(\frac{t}{\sqrt{1+m^2}}, \frac{mt}{\sqrt{1+m^2}} + b),\ P \in L_{m,b},\ f(P) = t]$

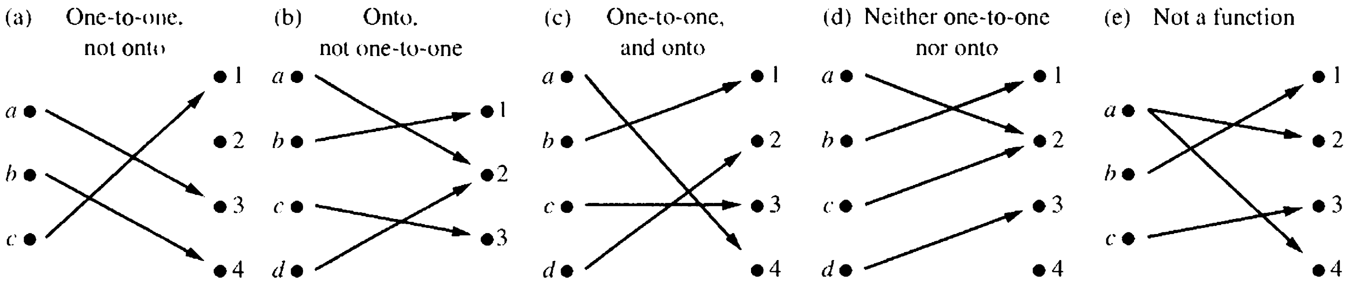
<u>Definition</u> (injection)

A function f is said to be one-to-one, or injective, if and only if $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of f. A function is said to be an injection if it is one-to-one.

Remark: A function f is one-to-one if and only if $f(x) \neq f(y)$ whenever $x \neq y$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition. Note that we can express that f is one-to-one using quantifiers as $\forall x\ \forall y\ (f(x) = f(y) \Rightarrow x = y)$ or equivalently $\forall x\ \forall y\ (x \neq y \Rightarrow f(x) \neq f(y))$, where the universe of discourse is the domain of the function.

<u>Definition</u> (bijection)

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.



Examples of Different Types of Correspondences.

<u>Definition</u> (inverse function)

Let f be a bijection from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

8. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote Poincaré plane, and let ${}_aL$ denote type I line. Lets define function $g : {}_aL \to \mathbb{R}$ on the following way $g(a,y) = \ln(y)$. Show that g is bijection, and determine inversion of g. $[g^{-1}(t) = (a, e^t)]$

9. Let p denote line from incidence geometry $\{\mathcal{S}, \mathcal{L}\}$, and let $f : p \to \mathbb{R}$ denote surjection for which: $|f(P) - f(Q)| = d(P,Q)\ \forall P,Q \in p$; where d denote distance function on $\mathcal{S}$. Show that f is bijection. $[f(P) = f(Q) \Rightarrow P = Q]$

<u>Definition</u> ($\sinh(t)$, $\cosh(t)$, $\tanh(t)$, $\operatorname{sech}(t)$)

We define the hyperbolic sine, hyperbolic cosine, hyperbolic tangent and hyperbolic secant on the following way

$$\sinh(t) = \frac{e^t - e^{-t}}{2}; \qquad \cosh(t) = \frac{e^t + e^{-t}}{2};$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}; \qquad \operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}}.$$

10. Show that for every value of $t \in \mathbb{R}$

(i) $[\cosh(t)]^2 - [\sinh(t)]^2 = 1$; (ii) $[\tanh(t)]^2 + [\operatorname{sech}(t)]^2 = 1$.

11. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote Poincaré plane, and let ${}_cL_r$ denote type II line. Lets define function $g : {}_cL_r \to \mathbb{R}$ on the following way: $f(x,y) = \ln(x - c + r) - \ln(y)$. Show that f is bijection, and determine inversion of f. $[\ln(x-c+r) - \ln(y) = t,\ e^t = \frac{x-c+r}{y},\ e^{-t} = -\frac{x-c-r}{y},\ y = r\operatorname{sech}(t),\ x - c = r\tanh(t)]$